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CHARACTERIZATION OF POSITIVE DEFINITE AND SEMIDEFINITE MATRICES--ETC(U)

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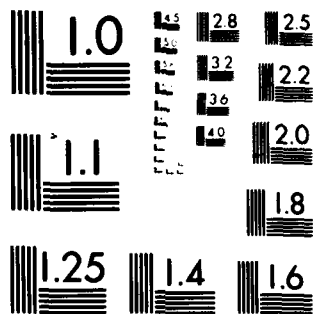
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CHARACTERIZATION OF POSITIVE DEFINITE  
AND SEMIDEFINITE MATRICES VIA QUADRATIC  
PROGRAMMING DUALITY

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ABSTRACT

Positive definite and semidefinite matrices induce well known duality results in quadratic programming. The converse is established here. Thus if certain duality results hold for a pair of dual quadratic programs, then the underlying matrix must be positive definite or semidefinite. For example if a strict local minimum of a quadratic program exceeds or equals a strict global maximum of the dual, then the underlying symmetric matrix  $Q$  is positive definite. If a quadratic program has a local minimum then the underlying matrix  $Q$  is positive semidefinite if and only if the primal minimum exceeds or equals the dual global maximum and  $x^T Q x = 0$  implies  $Qx = 0$ . A significant implication of these results is that the Wolfe dual may not be meaningful for nonconvex quadratic programs and for nonlinear programs without locally positive definite or semidefinite Hessians, even if the primal second order sufficient optimality conditions are satisfied.

AMS (MOS) Subject Classifications: 90C20, 15A63

Key Words: Positive definite matrices, quadratic programming,  
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## SIGNIFICANCE AND EXPLANATION

Constrained optimization problems occur in dual pairs when certain "valley" conditions (convexity or positive definiteness) are satisfied. This pairing has some very useful theoretical and computational implications. In this work we show the converse, namely that if a pair of problems are connected through a duality relation then these valley conditions must be satisfied.



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# CHARACTERIZATION OF POSITIVE DEFINITE AND SEMIDEFINITE MATRICES VIA QUADRATIC PROGRAMMING DUALITY

S.-P. Han & O. L. Mangasarian

## 1. INTRODUCTION

It is well known [3,4,11,10] that the dual quadratic programs

$$\begin{aligned}
 (1a) \quad & \underset{x}{\text{Minimize}} \quad \frac{1}{2}x^T Qx + p^T x \\
 & \text{subject to} \quad Ax \leq b \\
 & \quad \quad \quad Cx = d \\
 (1b) \quad & \underset{x,u,v}{\text{Maximize}} \quad -\frac{1}{2}x^T Qx - b^T u - d^T v \\
 & \text{subject to} \quad Qx + A^T u + C^T v + p = 0 \\
 & \quad \quad \quad u \geq 0
 \end{aligned}$$

where  $Q$ ,  $A$  and  $C$  are given real matrices of order  $n \times n$ ,  $m \times n$  and  $k \times n$  respectively, with  $Q = Q^T$ , and  $p$ ,  $b$  and  $d$  are given vectors in the real finite dimensional Euclidean spaces  $R^n$ ,  $R^m$  and  $R^k$  respectively, possess many important relations when  $Q$  is positive semidefinite or positive definite. In this paper we are interested in the converse: What sort of duality relations between (1a) and (1b) induce positive definiteness or semidefiniteness in  $Q$ ? A key role in deriving these converse relations is played by the following conjugate cone characterization of positive definite and semidefinite matrices [8].

**1.1 Theorem [8]** Let  $K$  be a nonempty convex polyhedral cone in  $R^n$ . The  $n \times n$  real matrix  $P$  is positive semidefinite if and only if  $P$  is positive semidefinite plus on the cone  $K$  and positive semidefinite on the conjugate cone  $K^P$ , that is

$$\begin{aligned}
 (2) \quad & x \in K \Rightarrow x^T P x \geq 0 \\
 & x^T P x = 0, x \in K \Rightarrow (P + P^T)x = 0
 \end{aligned}$$

$$(3) \quad y \in K^P := \{y | y(P+P^T)x \leq 0, \forall x \in K\} \Rightarrow y^T P y \geq 0$$

**1.2 Theorem [8]** Let  $K$  be a nonempty convex polyhedral cone in  $R^n$ . The  $n \times n$  real matrix  $P$  is positive definite if and only if  $P$  is positive definite on  $K$  and  $K^A$  that is:

$$0 \neq x \in K \Rightarrow x^T P x > 0$$

$$0 \neq y \in K^P \Rightarrow y^T P y > 0$$

With the help of these characterization theorems and the second order optimality conditions of quadratic programming [6,9,2,1] we show for example in Theorem 3.5 that if a strict local minimum of a quadratic program exceeds or equals a strict global maximum of the dual, then the matrix  $Q$  must be positive definite. In Theorem 3.6 we show that if a quadratic program has a local minimum then  $Q$  is positive semidefinite if and only if the primal minimum exceeds or equals the dual global maximum, and  $Qx = 0$  whenever  $x^T Q x = 0$ . In Corollary 3.7 we show that if the primal feasible and dual feasible sets are nonempty and if the weak duality relation holds, that is the primal objective exceeds or equals the dual objective over their respective feasible regions, and if  $Qx = 0$  whenever  $x^T Q x = 0$ , then  $Q$  is positive semidefinite. In [7] positive-definiteness of the Hessian of the Lagrangian of nonlinear programs was established under more restrictive assumptions.

The import of these and our other results is that when certain simple and desirable duality results are satisfied by a pair of dual quadratic programs, then the underlying matrix must be positive definite or semidefinite. This leads to the conclusion that the Dennis-Dorn-Wolfe quadratic dual programs [3,4,11] are meaningful only if the underlying matrix is positive definite or semidefinite. For example even if the primal quadratic problem (1a) has a unique global minimum solution (and thus satisfying the second order sufficient optimality condition), and if the underlying matrix is not positive semidefinite then the dual quadratic problem (1b) may not make sense. Thus the example

$$\text{minimize } x_1^2 - x_2^2 \quad \text{subject to } x_2 = 0$$

has the unique global solution  $x_1 = x_2 = 0$  but its dual

$$\text{maximize } x_1^2 - x_2^2 + vx_2 \quad \text{subject to } x_1 = 0, -2x_2 + v = 0$$

is unbounded above. Similarly the Wolfe dual for nonlinear programs may not be meaningful unless the Hessian of the Lagrangian is locally positive definite or semidefinite in the neighborhood of a stationary point of the primal problem [7]. Thus even if the second order sufficient optimality conditions are satisfied but the Hessian of the Lagrangian is not positive definite or semidefinite in a neighborhood of a local minimum solution, the dual problem may not make sense.

We shall need second order optimality conditions for the dual quadratic programs (1a) and (1b) which have local and strictly local solutions. These results can be found in [9,2,1] which we summarize here in a convenient form. The points  $(\bar{x}, \bar{u}, \bar{v}) \in R^{n+m+k}$  and  $(\bar{x}, \bar{u}, \bar{v}, \bar{w}) \in R^{n+m+k+n}$  are Karush-Kuhn-Tucker points of (1a) and (1b) respectively if they satisfy the following respective conditions [10]

|      |  |      |  |
|------|--|------|--|
| (4a) | $Q\bar{x} + A^T\bar{u} + C^T\bar{v} + p = 0$ | (4b) | $-Q\bar{x} + Q\bar{w} = 0$                   |
|      | $A\bar{x} \leq b$                            |      | $A\bar{w} - b \leq 0$                        |
|      | $C\bar{x} = d$                               |      | $C\bar{w} - d = 0$                           |
|      | $\bar{u} \geq 0$                             |      | $Q\bar{x} + A^T\bar{u} + C^T\bar{v} + p = 0$ |
|      | $\bar{u}^T(A\bar{x} - b) = 0$                |      | $\bar{u} \geq 0$                             |
|      |  |      | $\bar{u}^T(A\bar{w} - b) = 0$                |

Note that if  $(\bar{x}, \bar{u}, \bar{v})$  is a Karush-Kuhn-Tucker point of (1a) then  $(\bar{x}, \bar{u}, \bar{v}, \bar{x})$  is a Karush-Kuhn-Tucker point of (1b). To characterize local solutions we need to define the following index sets associated with a Karush-Kuhn-Tucker point  $(\bar{x}, \bar{u}, \bar{v})$  of (1a):



$$J := \{i | A_i \bar{x} = b_i, \bar{u}_i > 0\}$$

$$K := \{i | A_i \bar{x} = b_i, \bar{u}_i = 0\}$$

$$E := J \cup K = \{i | A_i \bar{x} = b_i\}$$

$$I := \{i | A_i \bar{x} < b_i, \bar{u}_i = 0\}$$

The notation  $A_E$  will represent the rows  $A_i$  of  $A$  with  $i \in E$ . We can now state the following.

**1.3 Theorem [2,1]** (Characterization of local solutions of quadratic programs) A point  $\bar{x} \in R^n$  is local minimum solution of the quadratic program (1a) if and only if  $\bar{x}$  and some  $(\bar{u}, \bar{v}) \in R^{m+k}$  satisfy the Karush-Kuhn-Tucker conditions (4a) and

$$(5a) \quad A_E x \leq 0, Cx = 0 \Rightarrow x^T Qx \geq 0$$

The Karush-Kuhn-Tucker point  $(\bar{x}, \bar{u}, \bar{v})$  of (1a) is a local maximum solution of the dual quadratic program (1b) if and only if

$$(5b) \quad Qx + A^T u + C^T v = 0, (u_K, u_I) \geq 0 \Rightarrow x^T Qx \geq 0$$

**1.4 Theorem [9,2,1]** (Characterization of strict local solutions of quadratic programs) A point  $\bar{x} \in R^n$  is a strict local minimum solution of the quadratic program (1a) if and only if  $\bar{x}$  and some  $(\bar{u}, \bar{v}) \in R^{m+k}$  satisfy the Karush-Kuhn-Tucker conditions (4a) and

$$(6a) \quad A_J x = 0, A_K x \leq 0, Cx = 0, x \neq 0 \Rightarrow x^T Qx > 0$$

The Karush-Kuhn-Tucker point  $(\bar{x}, \bar{u}, \bar{v})$  of (1a) is a strict local maximum solution of the dual quadratic program (1b) if and only if

$$(6b) \quad Qx + A^T u + C^T v = 0, u_K \geq 0, u_I = 0, (x, u, v) \neq 0 \Rightarrow x^T Qx > 0.$$

In the next two sections we characterize positive definite and semidefinite problems in terms of equality-constrained quadratic programs (Section 2) and inequality-constrained quadratic programs (Section 3). This split into equality- and

inequality-constrained problems permits the statement of somewhat sharper results for the former. For simplicity we confine the results of Section 3 to inequality constraints only. Problems with both equality and inequality constraints can be handled in a straightforward extension of the results of Section 3.

## 2. EQUALITY-CONSTRAINED QUADRATIC PROGRAMS

We specialize here the dual problems (1a) and (1b) to the following equality-constrained dual quadratic programs

$$\begin{array}{ll} (7a) \text{ Minimize } \frac{1}{2}x^T Qx + p^T x & (7b) \text{ Maximixe } -\frac{1}{2}x^T Qx - d^T v \\ \text{subject to } Cx = d & \text{subject to } Qx + C^T v + p = 0 \end{array}$$

We say that a problem is feasible, if the set of points satisfying its constraints is nonempty.

**2.1 Theorem** (Characterization of positive semidefinite and definite matrices) Let (7a) be feasible.

- (i) Let (7b) be feasible. A necessary and sufficient condition for  $Q$  to be positive semidefinite is that (7a) has a local minimum, (7b) has a local maximum solution and

$$(8) \quad x^T Qx = 0, Cx = 0 \Rightarrow Qx = 0$$

- (ii) A sufficient condition for  $Q$  to be positive definite is that (5a) has a strict local minimum solution and (5b) has a strict local maximum solution. This condition is also necessary if  $C$  has linearly independent rows.

**Proof** (i) Necessity follows from existence and duality theory of convex quadratic programming [5,10]. We establish sufficiency now by means of Theorem 1.1. Define

$$(9) \quad K := \{x | Cx = 0\}$$

Then

$$\begin{aligned} (10) \quad K^Q &:= \{y | y^T Qx \leq 0, \forall x \in K\} \\ &= \{y | y^T Qx > 0, Cx = 0 \text{ has no solution } x\} \\ &= \{y | Qy + C^T v = 0\} \end{aligned}$$

Since (7a) has a local minimum solution it follows by Theorem 1.3 (5a) and (9) that

$$(11) \quad x^T Q x \geq 0 \quad \text{for } x \in K$$

Since (7b) has a local maximum solution, it follows also by Theorem 1.3 (5a) and (10) that

$$(12) \quad y^T Q y \geq 0 \quad \text{for } y \in K^Q$$

Hence by (11), (8), (12) and Theorem 1.1,  $Q$  is positive semidefinite.

(ii) (Necessity) That both (7a) and (7b) have solutions follows from the feasibility of (7a) and the positive definiteness of  $Q$ . The uniqueness of solution for (7a) follows from the positive definiteness of  $Q$ . The uniqueness of solution for (7b) follows from the positive definiteness of  $Q$ , the linear independence of the rows of  $C$  and Theorem 1.4 (6b).

(Sufficiency) We establish sufficiency by means of Theorem 1.2. Since (7a) has a strict local minimum solution it follows by Theorem 1.4 (6a) and (9) that

$$(13) \quad x^T Q x > 0 \quad \text{for } 0 \neq x \in K$$

Since (7b) has a strict local maximum solution, it follows also by Theorem 1.4 (6a) that

$$x^T Q x > 0 \quad \text{for } Qx + C^T v = 0, (x, v) \neq 0$$

and hence by (10)

$$(14) \quad y^T Q y > 0 \quad \text{for } 0 \neq y \in K^Q$$

Hence by (13), (14) and Theorem 1.2,  $Q$  is positive definite.  $\square$

### 3. INEQUALITY-CONSTRAINED QUADRATIC PROGRAMS

We turn our attention now to the following inequality constrained dual quadratic programs

$$(15a) \text{ Minimize } \frac{1}{2} x^T Q x + p^T x$$

$$\text{subject to } A x \leq b$$

$$(15b) \text{ Maximize } -\frac{1}{2} x^T Q x - b^T u$$

$$\text{subject to } Cx + A^T u + p = 0$$

$$u \geq 0$$

**3.1 Theorem** (Characterization of positive semidefinite and definite matrices) Let (15a) be feasible.

- (i) Let (15b) be feasible. A necessary and sufficient condition for  $Q$  to be positive semidefinite is that (15a) has a local minimum solution  $\bar{x}$  with multiplier  $\bar{u}$ , that  $(\bar{x}, \bar{u})$  is a local maximum solution of (15b) and

$$(16) \quad x^T Q x = 0, A_J x = 0, A_K x \leq 0 \Rightarrow Qx = 0$$

- (ii) A sufficient condition for  $Q$  to be positive definite is that (15a) have a strict local minimum solution  $\bar{x}$  with multiplier  $\bar{u}$ , and  $(\bar{x}, \bar{u})$  is a strict local maximum solution of (15b). If in addition the rows of  $A_J$  are linearly independent,  $A_J x = 0, A_K x > 0$  has a solution, then this condition is also necessary.

**Proof** (i) Necessity follows from existence and duality theory of convex quadratic programs. We establish sufficiency now by means of Theorem 1.1. Define

$$(17) \quad K := \{x | A_J x = 0, A_K x \leq 0\} \subset \{x | A_E x \leq 0\}$$

Then

$$\begin{aligned} K^Q &= \{y | y^T Q x \leq 0, \forall x \in K\} = \{y | y^T Q x > 0, A_J x = 0, A_K x \leq 0, \text{ has no solution } x\} \\ &= \{y | Qy - A_J^T u_J - A_K^T u_K = 0, u_K \geq 0\} \end{aligned}$$

Therefore

$$(18) \quad -K^Q = \{x | Qx + A^T u = 0, u_K \geq 0, u_I = 0\}$$

Since  $\bar{x}$  is a local minimum solution of (15a) with multiplier  $\bar{u}$  it follows from Theorem 1.3 (5a) and (17) that

$$(19) \quad x^T Q x \geq 0 \quad \text{for } x \in K$$

Since  $(\bar{x}, \bar{u})$  is also a local maximum solution of (15b) it follows from Theorem 1.3 (5b) and (18) that  $x^T Q x \geq 0$  for  $x \in -K$  which is equivalent to

$$(20) \quad x^T Q x \geq 0 \quad \text{for } x \in K^Q$$

Conditions (19), (16), (20) and Theorem 1.1 imply that  $Q$  is positive semidefinite.

(ii) (Necessity) That both (15a) and (15b) have solutions follows from the feasibility of (15a) and the positive definiteness of  $Q$ . The uniqueness of solution of (15a) follows from the positive definiteness of  $Q$ . The uniqueness of solution of (15b) follows from the positive definiteness of  $Q$ , the linear independence of the rows of  $A_J$ , the existence of a solution to  $A_J x = 0$ ,  $A_K x > 0$  and Theorem 1.4 (6b).

(Sufficiency) We establish sufficiency by means of Theorem 1.2. Since (15a) has a strict local minimum solution  $\bar{x}$  it follows by Theorem 1.4 (6a) and (17) that

$$(21) \quad x^T Q x > 0 \quad \text{for } 0 \neq x \in K$$

Since  $(\bar{x}, \bar{u})$  is a strict local maximum solution of (15b) it follows from Theorem 1.4 (6b) and (18) that

$$(22) \quad x^T Q x > 0 \quad \text{for } 0 \neq x \in K^Q$$

Hence by (21), (22) and Theorem 1.2,  $Q$  is positive definite.  $\square$

### 3.2 Corollary (Globalization of local dual solutions)

- (i) Let  $\bar{x}$  be a local minimum solution of (15a) with multiplier  $\bar{u}$ , let  $(\bar{x}, \bar{u})$  be a local maximum solution of (15b) and let (16) hold. Then  $Q$  is positive semidefinite and hence  $\bar{x}$  is a global

minimum solution of (15a) and  $(\bar{x}, \bar{u})$  is a global maximum solution of (15b).

- (ii) Let  $\bar{x}$  be a strict local minimum solution of (15a) with multiplier  $\bar{u}$ , let  $(\bar{x}, \bar{u})$  be a strict local maximum solution of (15b) then  $Q$  is positive definite and hence  $\bar{x}$  is a global strict minimum solution of (15a) and  $(\bar{x}, \bar{u})$  is a global maximum solution of (15b).

We note that condition (16) of Theorem 3.1 cannot be dispensed as shown by the following pair of dual programs:

|  |  |
|--|--|
| <p>Minimize <math>x_1 x_2</math></p> <p>subject to <math>x_1 \geq 0</math></p> <p style="padding-left: 100px;"><math>x_2 \geq 0</math></p> | <p>Maximize <math>-x_1 x_2</math></p> <p>subject to <math>x_1 - u_2 = 0</math></p> <p style="padding-left: 100px;"><math>x_2 - u_1 = 0</math></p> <p style="padding-left: 100px;"><math>(u_1, u_2) \geq 0</math></p> |
|--|--|

Clearly  $(x_1, x_2) = (0, 0)$  is a global solution to the primal problem,  $(x_1, x_2, u_1, u_2) = (0, 0, 0, 0)$  is a Karush-Kuhn-Tucker point for the primal problem as well as a global solution to the dual problem. However the underlying matrix  $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  is not positive semidefinite because condition (16) is violated.

We establish now other duality results which induce positive definiteness or semidefiniteness. We begin by a few preliminary results.

**3.3 Lemma** Let  $(\bar{x}, \bar{u})$  satisfy the Karush-Kuhn-Tucker conditions of (15a). Then

$$(23) \quad \frac{1}{2} \bar{x}^T Q \bar{x} + p^T \bar{x} \geq -\frac{1}{2} x^T Q x - b^T u$$

implies that

$$(24) \quad -\frac{1}{2} \bar{x}^T Q \bar{x} - b^T \bar{u} \geq -\frac{1}{2} x^T Q x - b^T u$$

Proof From the Karush-Kuhn-Tucker conditions of (15a) we have that

$$-\bar{x}^T Q \bar{x} - p^T \bar{x} - b^T \bar{u} = 0$$

which when added to (23) yields (24).  $\square$

**3.4 Lemma** Let  $(\bar{x}, \bar{u})$  satisfy the Karush-Kuhn-Tucker conditions of (15a) such that for all  $(x, u)$  feasible for the dual quadratic program (15b)

$$(25) \quad \frac{1}{2} \bar{x}^T Q \bar{x} + p^T \bar{x} \geq -\frac{1}{2} x^T Q x - b^T u$$

Then  $(\bar{x}, \bar{u})$  solves (15b).

Proof Since  $(\bar{x}, \bar{u})$  is feasible for the dual quadratic program (15b) and since by (25) and Lemma 3.3

$$-\frac{1}{2} \bar{x}^T Q \bar{x} - b^T \bar{u} \geq -\frac{1}{2} x^T Q x - b^T u$$

for all dual feasible  $(x, u)$ , it follows that  $(\bar{x}, \bar{u})$  solves (15b).  $\square$

**3.5 Theorem** (Sufficient condition for positive definiteness) If a strict local minimum of the quadratic program (15a) exceeds or equals a strict global maximum of the dual quadratic program (15b) then  $Q$  is positive definite.

Proof Let  $\bar{u}$  be a multiplier associated with the strict local minimum solution of (15a). By Lemma 3.4,  $(\bar{x}, \bar{u})$  is a global maximum solution of (15b). By assumption this global maximum is strict. Hence by Theorem 3.1 (ii)  $Q$  is positive definite.  $\square$

**3.6 Theorem** (Characterization of positive semidefinite matrices) Let  $\bar{x}$  be a local minimum solution of (15a). The matrix  $Q$  is positive semidefinite if and only if (16) holds and for any dual feasible  $(x, u)$

$$(26) \quad \frac{1}{2} \bar{x}^T Q \bar{x} + p^T \bar{x} \geq -\frac{1}{2} x^T Q x - b^T u$$



Proof Necessity follows from the duality theory of quadratic programming and the fact that  $x^T Q x = 0$  implies  $Q x = 0$  for any symmetric positive semidefinite matrix. To prove sufficiency we note that there exists a  $\bar{u}$  such that  $(\bar{x}, \bar{u})$  is a Karush-Kuhn-Tucker point of (15a) and by (26) and Lemma 3.4,  $(\bar{x}, \bar{u})$  is a global maximum solution to (16b). Hence by Theorem 3.1 (i)  $Q$  is positive semidefinite.

A direct consequence of Theorem 3.6 is the following characterization of positive semidefinite matrices in terms of the weak duality [10] relation of quadratic programs.

3.7 Corollary (Positive semidefiniteness via weak duality)

Let the quadratic programs (15a) and (15b) be feasible. The matrix  $Q$  is positive semidefinite if and only if for all primal feasible  $x$  and all dual feasible  $(y, u)$

$$(27) \quad \frac{1}{2} x^T Q x + p^T x \geq -\frac{1}{2} y^T Q y - b^T u$$

and

$$(28) \quad z^T Q z = 0 \Rightarrow Q z = 0$$

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| 19. KEY WORDS (Continue on reverse side if necessary and identify by block number)<br><br>Positive definite matrices, quadratic programming, duality  |                                      |   |
| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br>Positive definite and semidefinite matrices induce well known duality results in quadratic programming. The converse is established here. Thus if certain duality results hold for a pair of dual quadratic programs, then the underlying matrix must be positive definite or semidefinite. For example if a strict local minimum of a quadratic program exceeds or equals a strict<br>(cont.) |                                      |   |

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ABSTRACT (cont.)

global maximum of the dual, then the underlying symmetric matrix  $Q$  is positive definite. If a quadratic program has a local minimum then the underlying matrix  $Q$  is positive semidefinite if and only if the primal minimum exceeds or equals the dual global maximum and  $x^T Q x = 0$  implies  $Qx = 0$ . A significant implication of these results is that the Wolfe dual may not be meaningful for nonconvex quadratic programs and for nonlinear programs without locally positive definite or semidefinite Hessians, even if the primal second order sufficient optimality conditions are satisfied.

$L(T)$